Canonical Reduction of Gravity: from General Covariance to Dirac Observables and post-Minkowskian Background-Independent Gravitational Waves.

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Abstract

The status of canonical reduction for metric and tetrad gravity in space-times of the Christodoulou-Klainermann type, where the ADM energy rules the time evolution, is reviewed. Since in these space-times there is an asymptotic Minkowski metric at spatial infinity, it is possible to define a Hamiltonian linearization in a completely fixed (non harmonic) 3-orthogonal gauge without introducing a background metric. Post-Minkowskian background-independent gravitational waves are obtained as solutions of the linearized Hamilton equations.

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In general relativity the metric tensor has a double role: it is the potential of the gravitational field and simultaneously describes the chrono-geometrical structure of the space-time in a dynamical way by means of the line element. It teaches to all the other fields relativistic causality: for instance it selects the paths to be followed by the rays of light in the geometrical optic approximation. This aspect of the metric tensor is completely lost every time it is split in a background metric plus a perturbation. This implies that, like in special relativity, we introduce a background non-dynamical (absolute) chrono-geometrical structure and the perturbation (like every other massless field) propagates along the fixed background light-cone. As a consequence, since we do not know realistic solutions of Einstein's equations for macroscopic bodies, the only existing theory of measurement in general relativity is the axiomatic one of Ehlers, Pirani and Schild [1], which utilizes *test* massive particles and rays of light in place of their *dynamical* counterparts. Moreover, we have that in the approaches to quantum gravity based on the introduction of a background metric (effective quantum field theory, string theory) gravity is simulated by gravitons with the same space-time behaviour as photons and gluons except for the spin and no method is know to rebuild the dynamical chrno-geometrical structure.

On the other hand the only sufficiently developed background-independent approach to quantum gravity, namely loop quantum gravity, gives rise to a suggestive quantum geometry for the 3-space but has problems with time evolution and does not correspond to a Fock space. As a consequence, it is not known how to introduce electro-magnetism (not to speak of the standard model of elementary particles) so to make a comparison with ordinary perturbative quantum electro-dynamics and understand its modifications induced by gravity.

This state of affairs motivated an attempt [2, 3, 4] to revisit classic metric gravity [2] and its ADM Hamiltonian formulation to see whether it is possible to define a model of general relativity able to incorporate fields and particles and oriented to a background-independent quantization. First of all to include fermions it is natural to resolve the metric tensor in terms of cotetrad fields [3, 4] [$g_{\mu\nu}(x) = E^{(\alpha)}_{\mu}(x)\, \eta_{(\alpha)(\beta)}\, E^{(\beta)}_{\nu}(x)$; $\eta_{(\alpha)(\beta)}$ is the flat Minkowski metric in Cartesian coordinates] and to reinterpret the gravitational field as a *theory of time-like observers endowed with tetrads*, whose dynamics is controlled by the ADM action thought as a function of the cotetrad fields. Since the standard model of elementary particles and its extensions are a chapter of the theory of representations of the Poincare' group on the non-compact Minkowski space-time and we look for a Hamiltonian decription, the mathematical pseudo-Riemannian 4-manifold M^4 introduced to describe space-time is assumed to be non-compact and globally hyperbolic. This means that it admits 3+1 splittings with foliations whose leaves are space-like Cauchy 3-surfaces assumed diffeomorphic to R^3 (so that any two points on them are joined by a unique 3-geodesic). As discussed in Ref.[5] these 3-surfaces are also *simultaneity surfaces*, namely a convention for the synchronization of clocks (in general different from Einstein's one, valid in inertial systems in special relativity). If τ is the mathematical time labeling these 3-surfaces, Σ_τ, and $\vec{\sigma}$ are 3-coordinates (with respect to an arbitrary observer, a centroid $x^\mu(\tau)$, chosen as origin) on them, then $\sigma^A = (\tau, \vec{\sigma})$ can be interpreted as Lorentz-scalar radar 4-coordinates and the surfaces Σ_τ are described by embedding functions $x^\mu = z^\mu(\tau, \vec{\sigma})$. In these coordinates the metric is $g_{AB}(\tau, \vec{\sigma}) = z^\mu_A(\tau, \vec{\sigma})\, g_{\mu\nu}(z(\tau, \vec{\sigma}))\, z^\nu_B(\tau, \vec{\sigma})$ $[z^\mu_A = \partial z^\mu / \partial \sigma^A]$. Since the 3-surfaces Σ_τ are *equal time* 3-spaces with all the clocks synchronized, the spatial distance between two equal-time events will be $dl_{12} = \int_1^2 dl\, \sqrt{{}^3g_{rs}(\tau, \vec{\sigma}(l))\, \frac{d\sigma^r(l)}{dl}\, \frac{d\sigma^s(l)}{dl}}$ $[\vec{\sigma}(l)$ is a parametrization of the 3-geodesic γ_{12} joining the two events on $\Sigma_\tau]$. Moreover, by using test rays of light we can define the *one-way* velocity of light between events on Σ_τ. Therefore, the Hamiltonian description has naturally built in the tools (essentially the 3+1 splitting) to make contact with experiments in a relativistic framework, where simultaneity is a frame-dependent property. The manifestly covariant description using Einstein's equations is the natural one for the search of exact solutions, but is inadequate to describe experiments.

Other requirements [3, 4] on the Cauchy and simultaneity 3-surfaces Σ_τ induced by particle physics are:

i) Each Σ_τ must be a Lichnerowitz 3-manifold [6], namely it must admit an involution so that a generalized Fourier transform can be defined and the notion of positive and negative frequencies can be introduced (otherwise the notion of particle is missing like it happens in quantum field theory in arbitrary curved space-times).

ii) Both the cotetrad fields (and the metric tensor) and the fields of the standard model of elementary particles must belong to the same family of suitable weighted Sobolev spaces so that simultaneously there are no Killing vector fields on the space-time (this avoids the cone-over-cone structure of singularities in the space of metrics) and no Gribov ambiguity (either gauge symmetries or gauge copies [7]) in the particle sectors; in both cases no well defined Hamiltonian description is available.

iii) The space-time must be *asymptotically flat at spatial infinity* and with boundary conditions there attained in a way independent from the direction (like it is needed to define the non-Abelian charges in Yang-Mills theory [7]). This eliminates the supertranslations (the obstruction to define angular momentum in general relativity) and reduces the *spi group* of asymptotic symmetries to the ADM Poincare' group. The constant ADM Poincare' generators should become the standard conserved Poincare' generators of the standard model of elementary particles when gravity is turned off and the space-time becomes the Minkowski one. As a consequence, the *admissible foliations* of the space-time must have the simultaneity surfaces Σ_τ tending in a direction-independent way to Minkowski space-like hyper-planes at spatial infinity, where they must be orthogonal to the ADM 4-momentum. But these are the conditions satisfied by the Christodoulou-Klainermann space-times [8] , which are near Minkowski space-time in a norm sense. Therefore the surfaces Σ_τ define the *rest frame* of the τ-slice of the universe and there are asymptotic inertial observers to be identified with the *fixed stars* (the standard origin of rotations to study the precession of gyroscopes in space). In this class of space-times there is an *asymptotic Minkowski metric* (asymptotic background), which allows to define weak gravitational field configurations *without splitting the metric* in a background one plus a perturbation and without being a bimetric theory of gravity.

The absence of supertranslations also implies that the lapse and shift functions have the form $N(\tau, \vec{\sigma}) = \epsilon + n(\tau, \vec{\sigma})$, $N_r(\tau, \vec{\sigma}) = n_r(\tau, \vec{\sigma})$ [$\epsilon = +1$ with signature (+—) and $\epsilon = -1$ with signature (-+++)], with $n(\tau, \vec{\sigma})$, $n_r(\tau, \vec{\sigma})$ tending to zero at spatial infinity.

In ADM tetrad gravity there are 16 configuration variables: the cotetrad fields can be parametrized in terms of $n(\tau, \vec{\sigma})$, $n_r(\tau, \vec{\sigma})$, 3 boost parameters $\varphi_{(a)}(\tau, \vec{\sigma})$, 3 angles $\alpha_{(a)}(\tau, \vec{\sigma})$ and cotriads $e_{(a)r}(\tau, \vec{\sigma})$ on Σ_τ [$a = 1, 2, 3$]. There are 14 first class constraints $\pi_n(\tau, \vec{\sigma}) \approx 0$, $\pi_{n_r}(\tau, \vec{\sigma}) \approx 0$, $\pi_{\varphi_{(a)}}(\tau, \vec{\sigma}) \approx 0$ (the generators of local Lorentz boosts), $M_{(a)}(\tau, \vec{\sigma}) \approx 0$ (the generators of local rotations), $\Theta^r(\tau, \vec{\sigma}) \approx 0$ (the generators of the changes of 3-coordinates on Σ_τ) and $\mathcal{H}(\tau, \vec{\sigma}) \approx 0$ (the superhamiltonian constraint).

It can be shown [2, 4] that the addition of the DeWitt surface term to the Dirac Hamiltonian (needed to make the Hamiltonian theory well defined in the non-compact case) implies that the Hamiltonian does not vanish on the constraint surface (no frozen reduced phase space picture in this model of general relativity) but is proportional to the *weak ADM energy* (i.e. its form as a volume integral), which governs the τ-evolution [9]. Moreover the Hamiltonian gauge transformations generated by the superhamiltonian constraint do not have the

Wheeler-DeWitt interpretation (evolution in local time), but transform an admissible 3+1 splitting into another admissible one (so that all the admissible notions of simultaneity are gauge equivalent). See Refs.[2, 4] for the scheme of addition of gauge fixings and, in particular, for the bibliography showing that a completely fixed Hamiltonian gauge becomes a unique 4-coordinate system on space-time only on the solutions of Einstein's equations (on-shell), so that it describes an extended physical laboratory. Finally it can be shown that Lichnerowicz's identification of the conformal factor of the 3-metric on Σ_τ ($\phi = [det\, {}^3g]^{1/12}$) as the unknown in the superhamiltonian constraint is the correct one: as a consequence the gauge variable describing the normal deformations of the simultaneity surfaces Σ_τ is the momentum $\pi_\phi(\tau, \vec{\sigma})$ canonically conjugate to $\phi(\tau, \vec{\sigma})$ (and not the trace of the extrinsic curvature of Σ_τ, the so called intrinsic York time).

In Ref.[10] there is a review of the various implications of Einstein's Hole Argument,namely of the property of general covariance. The invariance of Einstein's equations under active diffeomorphisms (the widest local symmetry group of general relativity, whose passive reinterpretation by Bergmann and Komar contains the ordinary coordinate transformations and the Legendre pull-back of the Hamiltonian gauge transformations) imply: i) the absence of determinism (only two of Einstein's equations contain dynamical information: four are restrictions on initial data and four are void due to Bianchi identities), i.e. the presence of arbitrary gauge variables; ii) absence of a physical individuation of the mathematical points as physical point-events of space-time. On one side such an individuation can be achieved by formulating four of the gauge fixing constraints as the requirement that a set of ordinary 4-coordinates coincides with four suitable scalar functions of the four eigenvalues of the Weyl tensor (Bergmann-Komar intrinsic pseudo-coordinates). This implies that in a sense the *space-time is the gravitational field itself* and that each 4-coordinate system has a *noncommutative structure* associated to it already at the classical level.

On the other side, following Dirac, we have to identify a canonical basis $r_{\bar{a}}(\tau, \vec{\sigma})$, $\pi_{\bar{a}}(\tau, \vec{\sigma})$, $\bar{a} = 1, 2$, of *Dirac observables* (DO) as the physical degrees of freedom of the gravitational field, i.e. a canonical basis of predictable gauge-invariant quantities satisfying deterministic Hamilton equations governed by the weak ADM energy. This can be achieved by means of a Shanmugadhasan canonical transformation adapted to 13 of the 14 first class constraints (not to the superhamiltonian one), which turns out to be a *point* canonical transformation as a consequence of the form of the finite gauge transformations. In the new canonical basis 13 new momenta vanish due to the 13 constraints and their 13 conjugate configuration variables are *abelianized gauge variables* (as already said the 14th one is π_ϕ). While the 14 gauge variables describe *generalized inertial effects*, the DO describe *generalized tidal effects*. Since the transformation is a point one, the old momenta are linear functionals of the new ones, with the kernels determined by a set of elliptic partial differential equations. In this way for the first time we can identify a canonical basis of non-local and non-tensorial DO, which remains canonical in the class of gauges where $\pi_\phi(\tau, \vec{\sigma}) \approx 0$, even if no one knows how to solve the superhamiltonian constraint, i.e. the Lichnerowicz equation for the conformal factor.

A special completely fixed Hamiltonian gauge in this class can be defined by adding other 13 suitable gauge fixing constraints. In this gauge, which turns out to be non-harmonic in the weak field regime, the 3-metric on Σ_τ is *diagonal* and it corresponds to a unique 3-orthogonal 4-coordinate system on space-time (with an associated admissible 3+1 splitting with well defined simultaneity leaves) on the solutions of Hamilton equations. In this gauge it is possible to give a *background-independent* definition of a weak gravitational field: the DO $r_{\bar{a}}(\tau, \vec{\sigma})$, $\pi_{\bar{a}}(\tau, \vec{\sigma})$ should be slowly varying on a wavelength of the resulting post-Minkowskian gravitational wave, with the configurational DO $r_{\bar{a}}$ replacing the two polarizations of the harmonic gauges. A *Hamiltonian linearization* is defined in the following way [11]:

i) Assuming $ln\, \phi(\tau, \vec{\sigma}) = O(r_{\bar{a}})$, the Lichnerowitz equation can be linearized and for the first time a non-trivial solution for ϕ can be found. Using this solution all the other constraints and the elliptic canonicity conditions can be linearized and solved. By putting these solutions in the integrand of the weak ADM energy, we get a well defined form for the energy density in this gauge in terms of the DO, i.e. the physical degrees of freedom of the gravitational field.

ii) The resulting ADM energy is approximated with the terms *quadratic* in the DO and the resulting linearized Hamilton equations are studied and solved. It is explicitly checked that the linerized Einstein's equations are satisfied by this solution. Even if the gauge is not harmonic, the wave equation $\Box r_{\bar{a}}(\tau, \vec{\sigma}) = 0$ is implied by the Hamilton equations and solutions satisfying the universe rest-frame condition are found (they cannot be transverse waves in the rest frame). These are the *post-Minkowskian background-independent gravitational waves*. The deformation patterns of a sphere of test particles induced by $r_{\bar{1}}$ and $r_{\bar{2}}$ are determined by studying the geodesic deviation equation.

We are now studying tetrad gravity coupled to a perfect fluid described by a suitable singular Lagrangian [12]. The Hamiltonian linearization in the special 3-orthogonal gauge, together with an adaptation to our formalism of the theory of Dixon's multipoles [13], will allow to find the post-Minkowskian (without any post-Newtonian approximation!) generalization of the quadrupole emission formula and the explicit form in this gauge of the action-at-a-distance Newton and gravito-magnetic potentials inside the fluid together with its tidal interactions. The resulting formalism should help to find a description of binary systems in a post-Minkowskian regime, where the post-Newtonian approximations fail. Moreover, the two-body problem of general relativity in the post-Minkowskian weak field regime will be studied by using a new semi-classical regularization of the self-energies, implying the $i \neq j$ rule like it appens in the electro-magnetic case [14]. Also tetrad gravity coupled to Klein-Gordon, electro-magnetic and Dirac fields is under investigation.

It will be explored the possibility of defining a scheme of Hamiltonian numerical relativity, based on expansions in the Newton constant G (the so called post-Minkowskian approximations), to study the strong field regime of tetrad gravity.

Finally we will try to find the Hamiltonian re-formulation of the Newman-Penrose formalism. This would allow to look for a Shanmugadhasan canonical basis in which the physical degrees of freedom of the gravitational field are described by *Bergmann observables* (BO), namely a set of coordinate-independent scalar DO. Such a basis would allow to start a new program of quantization of gravity, based on the idea of quantizing only the tidal effects (the BO) and not the inertial ones (the gauge variables), since the latter describe only the *appearances* of the phenomena. A prototype of this quantization is under study in special relativity to arrive at a formulation of atomic physics in non-inertial systems: while for relativistic particles (and their non-relativistic limit) there are already preliminary results [15], for the inclusion of the electro-magnetic field we have to find a way out from the Torre-Varadarajan no-go theorem, the obstruction to the Tomonaga-Schwinger formalism. Moreover, if the weak ADM energy in a completely fixed Hamiltonian gauge can be expressed in terms of BO, this would help to clarify the problem of the coordinate-dependence of the energy density in general relativity, which we think is a preliminary step for a correct understanding of the cosmological constant and dark energy problems.

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